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CONTENTS

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THE BOUNDARY OF A SIMPLY CONNECTED DOMAIN[1]

GEORGE PIRANIAN

1. Prime ends. The foundation for the study of boundaries of simply connected domains in the plane was laid by Carathéodory [2], who defined ends in general and prime ends in particular, and who classified prime ends into four kinds. To facilitate a brief survey of the subject of prime ends, I introduce a few definitions concerning a simply connected domain B. In my definitions, I follow essentially the work of Carathéodory, except that, for the sake of brevity, I omit the description of ends in general and aim directly at prime ends.

A sequence K of crosscuts c_n of B is a *chain* provided

(i) the diameter of c_n tends to 0 as $n \to \infty$;

(ii) for each index n, the set $\bar{c}_n \cap \bar{c}_{n+1}$ (where the bar indicates closure) is empty;

(iii) some fixed point O in B cannot be joined to any crosscut c_n $(n>1)$ by any path in B which does not meet the crosscut c_{n-1}.

Two chains $K = \{c_n\}$ and $K' = \{c'_n\}$ in B are *equivalent* provided each crosscut c_n effects a separation, relative to B, of the point O from all except finitely many of the crosscuts c'_n.

An equivalence class of chains in B is a *prime end* of B.

If P is a prime end of B, let $K = \{c_n\}$ denote a chain which belongs to P, and for each index n let B_n denote that subdomain of B which is determined by c_n and does not contain the point O. The set $I(P) = \bigcap \bar{B}_n$ will be called the *impression* of P.

It should be remarked that Carathéodory and some other writers applied the term *prime end* to the point set $I(P)$, but that they regarded as distinct two prime ends P_1 and P_2 corresponding to two nonequivalent chains, even in cases where the two sets $I(P_1)$ and $I(P_2)$ are identical. The distinction between a prime end and its impression formalizes the ideas which are involved.

Carathéodory's principal theorem on the correspondence between boundaries under conformal mappings [2, p. 350] can be expressed as follows: *If $f(z)$ maps the unit disk conformally and one-to-one onto the*

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domain B, it induces a one-to-one mapping between the points of the unit circle and the prime ends of B.

2. Alternate definitions, and extensions. Koebe [**10**, pp. 217–219] introduced an alternate definition of prime ends (boundary elements). It is based on equivalence classes of paths to accessible boundary points. Two boundary paths Γ_1 and Γ_2 in B (that is, two homeomorphic images $p_i = p_i(t)$ $(i=1, 2)$ of the open unit interval) are equivalent if they both approach the same boundary point $p(1)$ as $t \to 1$ and if, for each $\epsilon > 0$, every point $p_1(t)$ $(t > t_\epsilon)$ can be joined to Γ_2 by a path which lies in B and has diameter less than ϵ. Prime ends correspond to "Dedekind sections" in the space of equivalence classes of paths to accessible boundary points.

Ursell and Young [**16**, see (4.5) and (4.6) on p. 5] took an approach intermediate between that of Carathéodory and that of Koebe. Their principal contributions are the study of the fine structure of the individual prime end, and the investigation of the relation between prime ends of B and the interior of the complement of B.

It is natural that the concept of prime ends should be subjected to axiomatization, and that it should be extended to domains of infinite connectivity, to domains in euclidean three-space, and finally to abstract spaces. In this direction, papers by Kaufmann [**9**], Mazurkiewicz [**13**], and Freudenthal [**6**] must be mentioned. Suvorov [**15**] defined prime ends of a sequence of domains converging to a nucleus. But these are developments which lead away from my problem, and I return to the ideas of Carathéodory.

3. The classification of prime ends. A point p in the impression $I(P)$ is a *principal point* (relative to P) provided every neighborhood of p contains a crosscut of a chain which belongs to P; otherwise, it is a *subsidiary point* (relative to P). Elementary considerations show that every impression $I(P)$ contains at least one point which is a principal point relative to P.

A prime end is of the *first kind* if its impression consists of a single point, which is then necessarily a principal point. In other words, P is of the first kind provided

(a) $I(P)$ contains only one principal point, and

(b) $I(P)$ contains no subsidiary points.

A prime end is of the *second*, *third*, or *fourth* kind if it satisfies only condition (a), only condition (b), or neither of the two conditions, respectively.

For the sake of brevity, we shall also say that a point set on the boundary of B is an impression of the first, second, third, or fourth

kind if it is the impression of a prime end of the corresponding kind.

By way of illustration, we consider the domain suggested by Figure 1. The domain consists of the interior of a square from which various line segments have been deleted. Each impression on the upper edge AC of the square is of the first kind; the same is true of impressions lying on slits that issue from the upper edge. The point B is the impression of infinitely many prime ends.

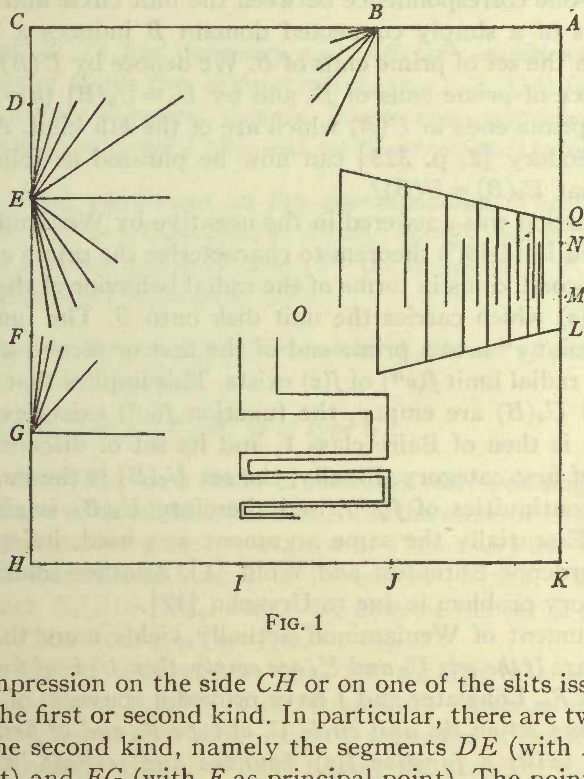

FIG. 1

Each impression on the side CH or on one of the slits issuing from CH is of the first or second kind. In particular, there are two impressions of the second kind, namely the segments DE (with D as principal point) and EG (with F as principal point). The point sets consisting of E and G, respectively, are the impressions of infinitely many prime ends of the first kind.

On HK, there are two impressions of the second kind, namely the segment IJ with I as principal point, and the same segment with J as principal point. In addition, the segment IJ is an impression of the third kind. If the figure is modified by adjoining infinitely many further slits (following closely the slits that approach the segment IJ), then IJ becomes the impression of infinitely many prime ends of the third kind.

On KA and the adjoining slits, all impressions are of the first kind, except that the segment LQ is an impression of the fourth kind. The points on the closed segment MN are principal points; the points on the remainder of the closed segment LQ are subsidiary points. The two point sets consisting of L and Q, respectively, are impressions of the first kind.

4. The distribution problem. Urysohn observed [17, p. 235] that the one-to-one correspondence between the unit circle and the set of prime ends of a simply connected domain B induces a Hausdorff topology in the set of prime ends of B. We denote by $U(B)$ the topological space of prime ends of B, and by $U_k = U_k(B)$ $(k=1, 2, 3, 4)$ the set of prime ends in $U(B)$ which are of the kth kind. A question of Carathéodory [2, p. 325] can now be phrased as follows: Is it possible that $U_2(B) = U(B)$?

This question was answered in the negative by Weniaminoff [18], who applied Lindelöf's theorem to characterize the prime ends of the first and second kinds in terms of the radial behavior of the mapping function $f(z)$ which carries the unit disk onto B. The function $f(z)$ carries a point $e^{i\theta}$ into a prime end of the first or second kind if and only if the radial limit $f(e^{i\theta})$ of $f(z)$ exists. This implies that if the sets $U_3(B)$ and $U_4(B)$ are empty, the function $f(e^{i\theta})$ exists everywhere. Also, $f(e^{i\theta})$ is then of Baire class 1, and its set of discontinuities is therefore of first category. Finally, the set $U_2(B)$ is the image of the set of discontinuities of $f(e^{i\theta})$, and therefore $U_2(B)$ is also of first category. Essentially the same argument was used, independently, by van Aardenne-Ehrenfest and Wolff [1]. Another solution of the Carathéodory problem is due to Urysohn [17].

The argument of Weniaminoff actually yields more that I have stated so far: *If the sets U_3 and U_4 are empty, then U_2 is of first category and of type F_σ.* Lohwater and I have proved a converse of this [12]: *Let E_2 be any set on the unit circle C, of type F_σ and of first category. Then there exists a function $f(z)$, bounded and schlicht in D, which carries each point of E_2 into a prime end of the second kind, and each point of $C-E_2$ into a prime end of the first kind.*

The fact that Carathéodory's special question on the set U_2 is amenable to such complete treatment constitutes a temptation to propose the following problem.

PROBLEM 1. *To find necessary and sufficient conditions on the decomposition of the unit circle C into four disjoint sets E_k $(k=1, 2, 3, 4)$ in order that, under some conformal mapping of the unit disk onto a bounded schlicht domain, the points of each set E_k correspond to prime ends of the kth kind.*

The solution of this general problem will be vastly more difficult than the solution of the special problem where U_3 and U_4 are empty. The difference is not merely one of degree, but it goes very deep; for the solution of the special problem can be stated in terms of topological concepts alone, while Fatou's theorem [5, p. 337] implies that the point sets E_3 and E_4 are subject to metric restrictions. Pending the development of methods and skills which are not available today, it seems advisable to replace Problem 1 by a less forbidding version:

PROBLEM 2. *To find necessary and sufficient conditions on the decomposition of C into four disjoint sets E_k ($k=1, 2, 3, 4$) in order that, for some simply connected domain B and some appropriate homeomorphism between C and $U(B)$, each set E_k corresponds to the set $U_k(B)$.*

5. Necessary conditions on the decomposition. Throughout this section, B denotes a fixed (but arbitrary) simply connected domain; P denotes a prime end of B, and $|P|$ the diameter of P, defined as the diameter of its impression $I(P)$.

THEOREM 1. *For every positive number h, the set of prime ends of diameter at least h is closed.*

COROLLARY. *The set $U_1(B)$ is of type G_δ.*

PROOF. Let $\{B_n\}$ be a chain of domains corresponding to P, and suppose that every neighborhood of P (in the sense of Urysohn) contains a prime end of diameter at least h. Then every set $\overline{B}_n$ contains a subset of diameter at least h, and therefore $|P| \geqq h$. The corollary follows since $U_1(B) = \bigcap M_n$, where M_n denotes the set of prime ends of diameter less than $1/n$ ($n=1, 2, \cdots$).

For the further analysis, we need the concept of cluster sets. A complex number w belongs to the (complete) *cluster set* of the function $f(z)$ at $e^{i\theta}$ provided there exists a sequence of points z_n in the unit disk D, converging to $e^{i\theta}$, and having the property that $f(z_n) \to w$ as $n \to \infty$. The point w belongs to the *radial cluster set* of $f(z)$ at $e^{i\theta}$, if the points z_n can be chosen on the radius of $e^{i\theta}$.

If f maps the unit disk D conformally onto B in such a way that the center of D is carried to the point O in B, and if $e^{i\theta}$ is the point corresponding to the prime end P with the chain $\{c_n\}$, then the image of the radius of $e^{i\theta}$ meets each of the cuts c_n. It follows that every principal point of P belongs to the radial cluster set of f at $e^{i\theta}$. On the other hand, Lindelöf has shown [11, p. 28] (see also Montel [14, pp. 48–52] and Gross [7, p. 254]) that the radial cluster set at $e^{i\theta}$ contains no subsidiary points of P. It follows that all points $e^{i\theta}$

at which the radial limit exists correspond to prime ends of the first or second kind, and that all points $e^{i\theta}$ at which the radial cluster set coincides with the complete cluster set correspond to prime ends of the first or third kind.

THEOREM 2. *On every interval of* $U(B)$, *the set* $U_1 \cup U_2$ *has the power of the continuum.*

This proposition follows immediately from Fatou's theorem. It can also be proved directly by showing that every interval of impressions of prime ends contains $2^{\aleph_0}$ points that are appropriately accessible from the interior of B.

THEOREM 3. *The set* $U_3 \cup U_4$ *is of type* $G_{\delta\sigma}$.

The theorem follows from the fact that the set of radii on which f has positive oscillation is a set of type $G_{\delta\sigma}$ (see Hausdorff [8, p. 273]).

Before closing this section, we call attention to a recent result of Collingwood [3] which implies that if the function f is meromorphic in D, then there exists a residual set E on C such that, at each point $e^{i\theta}$ in E, the radial cluster set at $e^{i\theta}$ coincides with the complete cluster set at $e^{i\theta}$. It follows (see Collingwood [4, p. 349]) that *for every simply connected domain* B *the set* $U_1 \cup U_3$ *is residual.*

6. **The case where U_4 is empty.** The results of the preceding section, together with an appropriate geometrical construction, lead to the following partial solution of the second distribution problem.

THEOREM 4. *Let* B *be a simply connected domain without prime ends of the fourth kind. Then there exist two sequences of sets* F_n *and* M_n *in the space* $U(B)$ *which have the following three properties.*

1. *The sets* F_n *are mutually disjoint, and the set* $\bigcup_{n=1}^{m} F_n$ *is closed, for every* m.

2. *For each* n, *the set* M_n *is of type* G_δ *and is contained in* F_n; *also, on every open subset of* F_n, *both* M_n *and its complement have the power of the continuum.*

3. $U_1 = U - \bigcup F_n$, $U_2 = \bigcup (F_n - M_n)$, $U_3 = \bigcup M_n$.

On the other hand, let $\{F_n\}$ *and* $\{M_n\}$ *be two sequences of sets on* C *such that Conditions* 1 *and* 2 *are satisfied, and let*

$$E_1 = C - \bigcup F_n, \qquad E_2 = \bigcup (F_n - M_n), \qquad E_3 = \bigcup M_n.$$

Then there exists a domain B *such that some homeomorphism between* C *and* $U(B)$ *carries each of the sets* E_k *into the corresponding set* $U_k(B)$ $(k = 1, 2, 3)$.

To prove the first part of the theorem, we let F_1 denote the set of

prime ends P with $|P| \geqq 1$, and for $n=2, 3, \cdots$ we let F_n denote the set of prime ends P with $1/(n-1) > |P| \geqq 1/n$. Condition 1 and the first part of Condition 3 are then satisfied, by Theorem 1.

Let f denote a function which maps the unit disk D conformally onto B, and let E_{nj} be the set of points $e^{i\theta}$ for which there exist two values r_1 and r_2 $(1-2^{-j} < r_1 < r_2 < 1)$ such that

$$|f(r_1 e^{i\theta}) - f(r_2 e^{i\theta})| > 1/n - 1/2^j.$$

Then E_{nj} is open, and therefore the set $E_n = \bigcap_{j=1}^{\infty} E_{nj}$ is of type G_δ. Now let M_n^* be the set of prime ends of B which correspond to points in E_n. Then each prime end in M_n^* lies in $\bigcup_{m=1}^{n} F_m$, by virtue of its diameter. Since E_n consists of all points of C on whose radius the function f has oscillation of amplitude at least $1/n$, and since U_4 is empty (by hypothesis), M_n^* consists of all prime ends of the third kind and of diameter at least $1/n$.

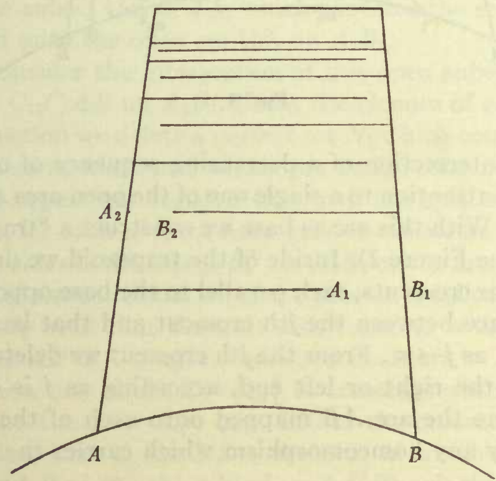

FIG. 2

Let $M_n = M_n^* \cap F_n$. The first part of Condition 2 is clearly satisfied; the second part follows from the theorem of Weniaminoff and the fact that a dense set of type G_δ is not denumerable.

7. The Carathéodory boxes. To complete the proof of the theorem, we use a device which is a slight modification of a construction described by Carathéodory [2; see the figure on p. 366 and the discussion on p. 369].

Let $\{F_n\}$ and $\{M_n\}$ be two sequences of point sets on C, subject to Conditions 1 and 2 of Theorem 4. Then each set M_n can be repre-

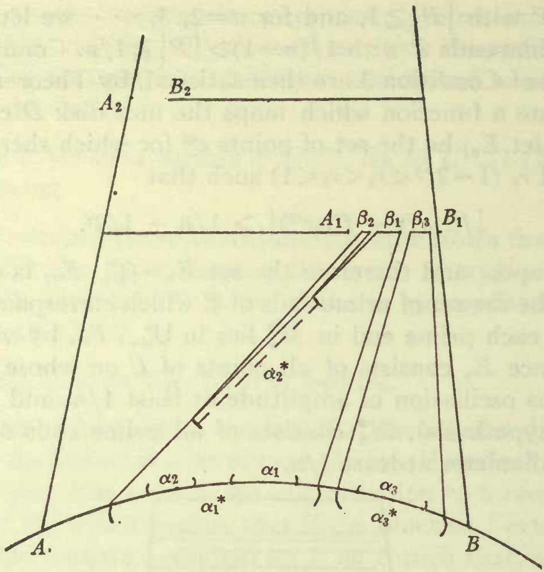

FIG. 3

sented as the intersection of a decreasing sequence of open sets G_{nk}. We confine our attention to a single one of the open arcs AB constituting the set G_{11}. With this arc as base we construct a "trapezoid" lying outside of C (see Figure 2). Inside of the trapezoid we draw infinitely many rectilinear crosscuts, each parallel to the base opposite AB, and with the distance between the jth crosscut and that base tending to 0 decreasingly, as $j \to \infty$. From the jth crosscut we delete a short segment A_jB_j at the right or left end, according as j is odd or even. And we imagine the arc AB mapped onto each of the deleted segments A_jB_j, by any homeomorphism which carries the point A into A_j.

By Condition 2 in the theorem, there exists in the closure of AB a perfect set which lies in the complement of M_1 and which contains both the points A and B. We choose such a set and denote it by N_0. Without loss of generality, we may assume that each of the open arcs of AB contiguous to N_0 contains at least one point of M_1; for any arc which fails in this respect can simply be incorporated into N_0.

Let $\alpha_1, \alpha_2, \cdots$ denote the open arcs that compose the set $AB - N_0$. We erect a trapezoid which has for one side the image β_1 of α_1 on A_1B_1, and for the opposite side an arc α_1^* on AB which contains α_1 and whose end points are essential points of N_0 (see Figure 3; by an essential point of N_0 we mean a point of N_0 which is not an

end point of any arc of the complement of N_0). We imagine the two segments of α_1^* which are complementary to α_1 mapped onto the corresponding sides of the quadrilateral which has just been erected.

If the arc α_2 does not lie on α_1, we accord it similar treatment. If it lies on α_1, we give it a quadrilateral one of whose sides is the image β_2 of α_2 on A_1B_1, while the opposite side is a segment α_2^* on the corresponding side of the quadrilateral over α_1; the segment α_2^* must contain the image of α_2, and its endpoints must be images of essential points of N_0.

We continue thus, with the proviso that no segment α_k^* shall have in its interior or as one of its endpoints any vertex of a previously constructed quadrilateral.

After each of the arcs α_k has been treated in this manner, we delete the arcs α_k^*. Now the unit disk, together with certain extensions that reach to the segment A_1B_1, constitutes a simply connected domain. And the open subset $\bigcup \alpha_k$ of G_{11}, which contains the set $M_1 \cap AB$, has been mapped onto the open set $\bigcup \beta_k$ on A_1B_1.

Next we consider the intersection of this open subset of A_1B_1 with the image of $G_{12} \cap AB$ on A_1B_1. From the closure of each component of this intersection we delete a perfect set N_1 which contains both endpoints of the component and whose pre-image does not meet the set M_1. We then repeat our construction with the open arcs $\alpha_k^{(1)}$ that remain; that is, for each of these open arcs we construct a quadrilateral one of whose sides is the image $\beta_k^{(1)}$ on A_2B_2 of the arc.

The result of the continuation of the process is called a *Carathéodory box*. It is easily seen that the unit disk together with the Carathéodory box over AB is a simply connected domain.

Consider now any chain $K = \{c_n\}$ of crosscuts which have at least one end in the Carathéodory box. Either the crosscuts c_n approach the upper edge, that is, the lid of the box, in which case K belongs to a prime end P of the third kind, and $I(P)$ coincides with the lid; or the crosscuts approach a point which is not on the lid. To analyze the latter case, we observe that every quadrangle which is erected during the construction is ultimately supplied with extensions which reach to the lid of the box; that every point which is a vertex of a quadrilateral is a limit point of such vertices; and that every vertex of a quadrilateral is a boundary point of the domain. If a chain of crosscuts approaches a point p which is a limit point of vertices of quadrilaterals and does not lie on the lid, then the chain belongs to a prime end P of the second kind; p is the principal point of P; and $I(p)$ consists of an infinite "polygonal line" and of the lid of the box.

It follows that there exists a homeomorphism between the unit

circle and the space of prime ends of our domain which maps into prime ends of the third kind all points on $M_1 \cap AB$; into prime ends of the second kind all limit points of this set which do not belong to the set; and into prime ends of the first kind all the remaining points of the circle.

It is easily seen how the same process can be applied to all other arcs of G_{11}. If the continuation is effected with uniformly high Carathéodory boxes, then the result is a domain which has prime ends of the third kind corresponding to all points of M_1, and prime ends of the second kind corresponding to all points of $\overline{M}_1 - M_1$.

Meanwhile, the open circular arcs of $C - \overline{M}_1$ have been transformed into open continuous arcs. On these arcs we can erect "parallel quadrilaterals" (in the manner of [**12**, §2]) in such a way that all points of $F_1 - M_1$ correspond to prime ends of the second kind.

The set $C - F_1$ is open, and its image consists of open continuous arcs. On these arcs we construct Carathéodory boxes corresponding to the set M_2, and further quadrilateral extensions corresponding to the set $F_2 - \overline{M}_2$. The Carathéodory boxes and the extensions may be distorted; but they can be made uniformly large, except near the impressions of prime ends corresponding to points of F_1. The indefinite continuation of the process presents no difficulty; for after the completion of each stage, all further modifications have to be made on open, continuous arcs of the boundary. And while the process would be certain to drive any draftsman to despair, it yields a domain B which constitutes a proof of the second part of Theorem 4.

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UNIVERSITY OF MICHIGAN

THE NOVEMBER MEETING IN LOS ANGELES

The five hundred thirty-ninth meeting of the American Mathematical Society was held at the University of California in Los Angeles on Friday and Saturday, November 15 and 16, 1957. There were 216 registrants, including 182 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, a Symposium on Game Theory was held on Friday. The program committee for the Symposium consisted of Professors David Blackwell, Chairman, H. F. Bohnenblust, Samuel Karlin, and Harold Kuhn. For the first session of the Symposium, Professor Blackwell presided, and half-hour talks were presented as follows: *Survey of continuous games on the unit square and outstanding problems*, Professor Samuel Karlin, Stanford University; *Infinitely repetitive games*, Dr. Philip Wolfe, RAND Corporation; *Multimove infinite games*, Dr. Melvin Dresher, RAND Corporation; *On multimove games*, Professor Rodrigo Restrepo, University of British Columbia; *Some experiments on games on electronic computing machines*, Dr. Stanislaw Ulam, Los Alamos Scientific Laboratory.

The second session of the Symposium was presided over by Dr. Wolfe, the talks being as follows: *A survey of extensive games*, Professor Harold Kuhn, Bryn Mawr College; *Markov learning models for game situations*, Professor Patrick Suppes, Stanford University. Chairman for the third session was Dr. Olaf Helmer, and the speakers were as follows: *Games with vector outcomes*, Dr. Lloyd Shapeley, RAND Corporation; *Simple games and finite solutions*, Dr. Herbert Gurk, Radio Corporation of America; *Descriptive theory of simple games*, Dr. John Isbell, University of Washington; *A cooperative n-person bargaining game*, Professor David Gale, RAND Corporation.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Harold Levine of Stanford University addressed the Society on Saturday, on the topic *Recent developments in the theory of wave motion*. He was introduced by Professor Arthur Erdelyi.

The sessions for contributed papers were presided over by Professors Charles Loewner, J. D. Swift, and Lida Barrett. Mr. Iwao Sugai was introduced by Professor J. W. Green and Mrs. Emma Lehmer by Professor V. L. Klee, Jr.

Abstracts of the contributed papers will be published in the February and April issues of the *Notices* of the American Mathematical Society.

V. L. KLEE, JR.,
Associate Secretary

THE NOVEMBER MEETING IN CORAL GABLES

The five hundred fortieth meeting of the American Mathematical Society was held at the University of Miami in Coral Gables, Florida, on Friday and Saturday, November 29–30. About 80 persons registered, including 65 members of the Society.

By invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings, there were two hour addresses. Professor Hans Rohrbach of the University of Mainz (visiting Professor at the University of North Carolina) spoke Friday evening on *Mathematical methods of ciphering and deciphering*, and Professor A. H. Clifford of Newcomb College, Tulane University, spoke Saturday morning on *Totally ordered commutative semigroups*. Professors Alfred Brauer and D. D. Miller presided at these sessions.

There were four sessions for contributed papers, Professor Gerald Huff, Donald Austin, Hans Rohrbach and J. H. Roberts presiding. Papers with abstract numbers 540-2, 540-21, 540-25, 540-27 and 540-28 were presented by title. Numbers 540-9, 540-13, 540-22 and 540-30 were presented, respectively, by A. T. Brauer, J. E. Maxfield, M. L. Curtis, and Naoki Kimura. Mr. Hunter was introduced by Professor R. J. Koch, Mr. Loeb by Dr. A. A. Goldstein, and Mr. Glasser by Professor Wayman Strother. Abstracts of the contributed papers will appear in the February and April issues of *Notices* of the American Mathematical Society.

JOHN H. ROBERTS,
Associate Secretary

THE NOVEMBER MEETING IN COLUMBIA

The five hundred forty-first meeting of the American Mathematical Society was held at the University of Missouri, Columbia, Missouri, on Friday and Saturday, November 29 and 30, 1957. There were about 75 registered including 60 members of the Society.

The Committee to Select Hour Speakers for Western Sectional Meetings had invited Professor Daniel Zelinsky of Northwestern University to address the Society. He spoke on the topic *Homological algebra* in Room 100, Stewart Hall, at 2:00 P.M. Friday. The presiding officer was Professor J. L. Zemmer.

There were two sessions for the presentation of contributed papers, one at 3:15 P.M. on Friday with Professor L. M. Kelly presiding, and the other at 10:00 A.M. Saturday with Professor W. G. Leavitt in charge.

Members of the Society and their friends were entertained at a tea held in the Student Union immediately following the sessions on Friday afternoon. An informal dinner was held in one of the restaurants on Friday night.

Abstracts of the contributed papers will be published in the February and April issues of the *Notices* of the American Mathematical Society.

J. W. T. YOUNGS,
Associate Secretary

RESEARCH PROBLEMS

1. Paul Slepian: *Problems on polynomials.*

(1) Let $0<A<1$. Let B be the set of all positive integers n such that there exist n positive numbers $a_1, a_2, \cdots, a_n$ such that the polynomial

$$(x^2 - 2Ax + 1) \prod_{i=1}^{n} (x + a_i)$$

has all non-negative coefficients. It is known that B is nonempty. (See P. M. Lewis, *The concept of the one in voltage transfer synthesis*, IRE Trans. Vol. CT-2, pp. 316–319, December, 1955.) Find the least element of B.

(2) Let $0<A<1$ and let N be the smallest integer in B, as described in (1) above. Does there exist $b>0$ such that

$$(x^2 - 2Ax + 1)(x + b)^N$$

has all non-negative coefficients?

(3) Generalize the questions raised in (1) and (2) above to the case where the factor $x^2-2Ax+1$ is replaced by an arbitrary real polynomial, say $\sum_{i=0}^{m} C_i x^i$, having no positive real roots. (Received September 13, 1957.)

2. Louis Weinberg: *Decomposition of Hurwitz polynomials.*

Let $q(s) = \sum_{k=0}^{n} a_k s^k$ represent a Hurwitz polynomial with real coefficients, i.e., all of its zeros have negative real parts. Can $q(s)$ be divided into the arithmetic sum of two polynomials,

$$q(s) = q_1(s) + q_2(s)$$

each of which has positive coefficients and only nonpositive real roots? This can easily be done in particular cases; for example, if $q(s) = (s^2+2s+5)(s+4) = s^3+6s^2+13s+20$, then $q_1(s) = s^3+6s^2+11s+6 = (s+1)(s+2)(s+3)$ and $q_2(s) = 2s+14$. If this can be shown to be impossible in the general case, can the decomposition always be carried out with three polynomials,

$$q(s) = q_1(s) + q_2(s) + q_3(s),$$

each of which again has positive coefficients and only nonpositive real roots? (Received September 19, 1957.)

3. R. E. Bellman: *Number theory.* I.

The problem of generating the integer solutions of the equation $x^2+y^2=1 \pmod p$ by means of the formula $x_n = \cos n\theta$, $y_n = \sin n\theta$, where (x_1, y_1) is a fundamental solution which we can write symbolically in the form $x_1 = \cos\theta_1$, $y_1 = \sin\theta_1$, has been extensively studied. What are the corresponding results for the equations $x_1^2+x_2^2+\cdots x_n^2 = 1 \bmod p$?

In particular, for the equation $x_1^2+x_2^2+x_3^2=1 \bmod p$, what subset of solutions do we obtain by means of the formulas

$$x_1 = \cos k\theta_1 \cos l\theta_2,$$
$$x_2 = \cos k\theta_1 \sin l\theta_2,$$
$$x_3 = \sin k\theta_1,$$

where $k, l = 0, 1, \cdots$, and θ_1, θ_2 correspond to certain primitive solutions? (Received May 22, 1957.)

4. R. E. Bellman: *Number theory.* II.

Consider the same type of problem for the multiplicative form

$$x^3 + y^3 + z^3 - 3xyz$$

and for the circulant functions of higher order. (Received May 22, 1957.)

5. R. E. Bellman: *Number theory.* III.

Consider the equation $y^2 = 4x^3 - g_2x - g_3$ which may be uniformized by means of the Weierstrassian elliptic functions $x = p(u)$, $y = p'(u)$. What subset of solutions of the congruence $y^2 = 4x^3 - g_2x - g_3 \pmod{p}$ can be obtained by means of the formulas $x = p(mu + nv)$, $y = p'(mu + nv)$, $m, n = 0, 1, 2, \cdots$, (not both zero simultaneously), where u and v correspond to certain primitive solutions?

Consider the similar problem for $y^2 = (1 - x^2)(1 - k^2x^2)$ which can be uniformized by means of Jacobian elliptic functions. (Received May 22, 1957.)

6. R. E. Bellman: *Number theory.* 1.

Let x be an irrational number in $[0, 1]$ and let $g(y; a, b)$, $0 \leqq a < b \leqq 1$ be a periodic function of y with period 1 defined by the conditions $g(y; a, b) = 1$, $a \leqq y \leqq b$, $g(y; a, b) = 0$ elsewhere for y in $[0, 1]$. Define the function

$$f_N(z, x) = g(z; a, b) + g(z + x; a, b) + \cdots + g(z + Nx; a, b)$$

for $1 \geqq z \geqq 0$, equal to the number of elements of the finite sequence $\{nx + z\}$, $n = 0, 1, \cdots, N$, falling inside $[a, b]$, modulo one.

The Weyl equidistribution theorem asserts that $f_N(z, x)/(N+1) \sim b - a$ as $N \to \infty$. It is easy to show via Fourier series that

$$\int_0^1 \int_0^1 [f_N(z, x) - (N + 1)(b - a)]^2 dz dx \sim (N + 1)c_1(a, b)$$

as $N \to \infty$.

This suggests that the quantity

$$u_N(z, x) = \frac{f_N(z, x) - (N + 1)(b - a)}{(N + 1)^{1/2}}$$

possesses asymptotic moments of all orders.

Does

$$\lim_{N\to\infty} \int_0^1 \int_0^1 \left(\frac{f_N(z, x) - (N + 1)(b - a)}{(N + 1)^{1/2}}\right)^{2k} dz dx,$$

$k = 2, 3, \cdots$ exist, and if so what is the limiting distribution? (Received July 15, 1957.)

7. R. E. Bellman: *Number theory.* 2.

Consider the same problem for the function

$$f_N(z, y, x) = g(z; a, b) + g(z + 2y + x; a, b) + \cdots + g(z + 2Ny + N^2x; a, b)$$

with x irrational, y and z in $[0, 1]$. As above, it is easy to show via Fourier series that

$$\int_0^1 \int_0^1 \int_0^1 [f_N(z, y, x) - (N + 1)(b - a)]^2 dx dy dz \sim (N + 1)c_1(a, b)$$

as $N \to \infty$.

Does

$$\lim_{N\to\infty}\int_0^1\int_0^1\int_0^1\left(\frac{f_N(z,y,x)-(N+1)(b-a)}{(N+1)^{1/2}}\right)^{2k}dxdzdy$$

exist for $k=1, 2, \cdots$, and if so, what is its value?

There are corresponding versions of this problem for polynomials of all orders. (Received July 15, 1957.)

8. R. E. Bellman: *Differential equations.*

Consider the second order linear differential equation $u''+(1+\lambda g(x))u=0$, where λ is a real constant and $\int_0^\infty |g(x)|\,dx<\infty$. Let $u_1(x)$ be the solution specified by $u_1(0)=0$, $u_1'(0)=1$. It is known that $u\sim r(\lambda)\sin(x+\theta(\lambda))$ as $x\to\infty$.

Taking λ to be complex variable, what are the analytic properties of the functions $r(\lambda)$ and $\theta(\lambda)$? In particular, where are the singularities nearest the origin?

If $g(x)>0$ for $x\geqq 0$, is the singularity nearest the origin on the negative axis? (Received August 16, 1957.)

REPORT OF THE TREASURER

The Treasurer this year again presents to the membership an abridged statement of the Society's financial position, set up in a semi-informal narrative style. A copy of the complete Treasurer's Report as submitted to the Trustees and the Council will be sent to any member requesting it from the Treasurer at the Providence office. Moreover, the Treasurer will be happy to answer any questions members may wish to put to him concerning the Society's financial affairs.

In general, it may be said that the Society's finances continue satisfactory, although the margin between income and expense is very narrow—indeed, our ordinary expenses exceeded our income by about $400, a very small overexpenditure of income.

Returns on invested funds this year have been at the rate of 4.65% computed on book value after deduction of custodial expense. This is slightly less than last year.

I

A DESCRIPTION OF THE FINANCIAL POSITION OF THE SOCIETY AS OF MAY 31, 1957

The Society had Cash on deposit		
In the Rhode Island Hospital Trust Co.	$ 81,363.74	
In various interest-bearing savings accounts	36,840.81	
In petty cash and drawing accounts in Providence and Los Angeles	1,100.00	
		$119,304.55
It had reserves invested until needed in Government bonds		54,313.04
There was owing to it		
By the United States Government	$ 25,168.17	
By members, subscribers and others	14,829.49	
		39,997.66
It had in stamps and in the postage meter		504.92
And had temporarily advanced to the Investment and other special accounts		4,555.97
Making a total of CURRENT ASSETS of		$218,676.14
The Society also held investment securities valued at		349,283.32
(The market value, May 31, 1957 was $445,907.00)		
TOTAL ASSETS, therefore, were		$567,959.46

Offsetting these assets, the Society			
Owed members, subscribers and vendors..	$4,548.59		
Held contributions designed for special purposes	1,191.82		
Making a total of Current Liabilities, money it might have to pay out on short notice, of		$ 5,740.41	
Held funds received from various special sources to support particular projects, such as the Summer Institute, the Register of Mathematicians, etc		78,931.59	
Had advanced for recovery from future sales for various Society publications—Colloquium and Survey volumes, Birkhoff papers, Russian translations, etc		30,941.23	
And held in its General Fund, the result of 68 years of prudent operations, the sum of		103,062.91	
Thus accounting for all the CURRENT FUNDS			$218,676.14
The Invested Funds represent the following:			
(1) The Endowment Fund, largely the gift of members about thirty years ago		$ 94,000.00	
(2) The Library Proceeds Fund, derived from the sale of the Society's Library in 1950		66,000.00	
(3) The Prize Funds—Bôcher, Cole, Moore		6,575.00	
(4) The Mathematical Reviews Fund, a gift received in 1940 to make possible the establishment of the Reviews		80,000.00	
(5) Reserves established by the Trustees to protect the life memberships and life subscriptions formerly available, and as a "hedge" against investment losses		72,130.23	
(6) Other funds, derived mainly from bequests to the Society by members which the Trustees were either required to invest or which they have invested at their option—the income being used for the general purposes of the Society		28,551.47	
A total of invested funds of		$347,256.70	
And this, together with cash due the current funds of..		2,026.62	
Accounts for the total holding of investment securities of			$349,283.32
TOTAL LIABILITIES, therefore, were			$567,959.46

II

An Account of the Financial Transactions of the Society During the Fiscal Year 1956–1957

The Society has two types of receipts—funds for special purposes and projects, and the General Fund, from which are met the general operating expenses of the organization, including the publication of the Bulletin, the Proceedings, and the Transactions. Income from sales of and subscriptions to these journals is placed in the General Fund, but in practice is allocated to the expenses of the journals themselves, as though such income were in fact special. It is so treated in the following presentation:

To meet its GENERAL obligations, the Society RECEIVED:

From dues and contributions of individual members	$67,365.32		
Less enlisted men's special discount	65.00		
		$ 67,300.32	
From dues of institutional members		24,814.00	
From dues of corporate members		3,000.00	
From investment and trusts		23,906.95	
From contracts, in payment of indirect costs		16,680.18	
From miscellaneous sources		3,722.64	
Total General Receipts			$139,424.09
These funds were EXPENDED			
For general administrative and meeting expenses		$ 64,475.33	
To meet deficits in Society publications:			
Bulletin: (Total expense, $21,263.67)	$14,848.93		
Proceedings: (Total expense, $35,527.40)	28,051.89		
Transactions: (Total expense, $38,119.40)	9,042.98		
Mathematical Reviews: (Total expense, $104,234.18)	6,382.08		
		$ 58,325.88	
In subsidies to non-Society publications		9,475.10	
To cover the Society's share of the cost of such joint projects as: the Policy Committee, the Combined Membership List, the Employment Register		4,566.50	
For miscellaneous expenses		2,954.29	
Total General Expenses			$139,797.10
Leaving an EXCESS OF EXPENSES OVER INCOME OF			$ 373.01
In addition, the Society incurred			
EXTRAORDINARY EXPENSES for moving and alterations			$ 16,834.13

Both of which were taken from moneys saved from previous years.

Respectfully submitted,
ALBERT E. MEDER, JR.
Treasurer

September 30, 1957

BOOK REVIEWS

An introduction to Diophantine approximation. By J. W. S. Cassels. Cambridge Tracts in Mathematics and Mathematical Physics, no. 35, New York, Cambridge University Press, 1957. 10+166 pp. $4.00.

The theory of Diophantine approximation is concerned with the approximate solution of Diophantine equations having no exact solutions, or only trivial ones. The first such problem considered was probably that of finding the good rational approximations to a real irrational number θ, or what is almost the same thing, of finding integers x, y, not both zero, for which $|\theta x - y|$ is small. This problem was generalized in the early 1840's by Dirichlet, who considered $|\theta_1 x_1 + \cdots + \theta_n x_n - y|$, and about 1850 by Hermite, who considered the system of quantities $|\theta_1 x - y_1|, \cdots, |\theta_n x - y_n|$. These papers, together with that of Liouville (1844) on the existence of transcendental numbers, might be regarded as the beginnings of the subject. Much later, Minkowski suggested the term "Diophantine approximation," and used it as title for the first book (1907) in the subject.

In 1936 J. F. Koksma's *Diophantische Approximationen* appeared in the Ergebnisse series. This beautiful work describes, usually without proofs, all the important research up to the time of publication, and contains an exhaustive bibliography. It is badly out of date now, of course, because the subject is one of the most active branches of number theory. A present day bibliography would probably contain at least twice as many references as the approximately 900 in Koksma's book. Strangely enough, the additional references would be nearly exclusively to papers by European and Russian authors; it is almost as if the subject were nonexistent as far as American mathematics is concerned.

Since Koksma's book, no general work has appeared until the tract under review, a fact which further increases the value of what is, in its own right, a very important piece of work. Cassels' book does not have the scope of Koksma's, and indeed it could not have under the space limitations imposed in the Cambridge series. No attempt is made to present a complete bibliography, nor to discuss all the important problems and topics in the field. (For example, just one paragraph is devoted to transcendental numbers.) Rather, the author presents a relatively small number of theorems, each with complete proof. Many of the most beautiful and significant results of the past 20 years are to be found, in most cases with new or simplified proofs

appearing here for the first time. Indeed, of the eight chapters only that on Roth's theorem is a rewrite of published work. The intending reader should be warned, however, that not all the economy of space comes about through genuine simplification; it is partly achieved by omissions of discussion, description and historical remarks, and by a terse style in the proofs themselves. No doubt space restrictions by the publishers made this necessary, and it is unquestionably better to have such excellent subject matter presented in somewhat condensed form than not at all. In any case, the book is not easy reading, and a beginner would be well advised to have pencil and paper at hand.

Chapter I, on homogeneous approximation, is concerned with systems of linear forms of the type mentioned above in connection with Dirichlet. In the simple case of a single form $|\theta x-y|$, there is available the immensely powerful theory of regular continued fractions, the usefulness of which derives from the fact that the convergents of the continued fraction expansion of θ are precisely the best approximations to θ. By a best approximation is meant a rational number p/q $(q>0)$ such that $|q\theta-p|<\|r\theta\|$ for all integers r with $0<r<q$. (Here $\|x\|$ represents the distance between x and the integer nearest x, so that $\|x\|=\min|x-n|$, the minimum extending over all integers n.) The author reverses the usual procedure; instead of defining a continued fraction and then verifying that its convergents have the desired property, he constructs the continued fraction as the solution of the problem of finding the best approximations. This is advantageous for generalization, as it points the way to the construction of the best approximations (from among the elements of a subfield) to an element of an arbitrary field of complex numbers, and it puts the emphasis on that aspect of continued fractions which is of greatest importance in number theory.

The remainder of the first chapter is devoted to the Dirichlet-Minkowski theorem, and a simple proof that that theorem is in a sense best possible.

Chapter II deals with the Markoff theory of the minima of indefinite binary quadratic forms. The proof presented goes back to works by Frobenius, Remak, C. A. Rogers and the author. Chapter III is concerned with inhomogeneous linear approximation, including Minkowski's theorem on the product of two linear forms, and Kronecker's theorem.

In Chapter IV on uniform distribution there is given a simple proof, from the definition, that the sequence $\{n\theta\}$ is uniformly distributed (mod 1) for irrational θ; the proof clearly generalizes to the

case of n-tuples of linear forms in m integral variables, if no nontrivial integral linear combination of the forms has integral coefficients. Weyl's criterion is then developed, and used to prove that if a polynomial $f(x)=a_r x^r+\cdots+a_0$ has at least one irrational coefficient a_j with $j>0$, then the sequence $\{f(n)\}$ is uniformly distributed (mod 1).

Chapter V contains a collection of important transference theorems (Übertragungssätze), by means of which information about one set of forms yields information about another set. The prototype of this class of theorems is due to Perron and Khintchine: Let $\theta_1, \cdots, \theta_n$ be irrational numbers, and let ω_1 and ω_2 be the respective upper bounds of the numbers ω, ω' such that the inequalities

$$\|u_1\theta_1+\cdots+u_n\theta_n\| \leqq (\max |u_j|)^{-n-\omega},$$
$$\max_{1\leqq j\leqq n} \|x\theta_j\| \leqq x^{-(1+\omega')/n}$$

have infinitely many integer solutions $(u_1, \cdots, u_n)$ and x. (ω_1 and ω_2 are nonnegative by the Dirichlet-Minkowski theorem.) Then

$$\omega_1 \geqq \omega_2 \geqq \frac{\omega_1}{n^2+(n-1)\omega_1}\cdot$$

More generally, the homogeneous approximation of a set of forms $\sum_i \theta_{ji}x_i$ can be related to that of the transposed set $\sum_j \theta_{ji}u_j$, and also to the inhomogeneous approximation of each of the sets. These theorems and the techniques involved in their proofs are used in the further consideration of some of the material of Chapter III.

Roth's remarkable improvement of the Thue-Siegel theorem is presented in Chapter VI. This theorem asserts that if ξ is an irrational algebraic number and δ is positive, then the inequality

$$\left|\xi-\frac{p}{q}\right|<\frac{1}{q^{2+\delta}}$$

has only finitely many solutions in integers $q>0$, p. The proof is succinct as usual, having shrunk to fifteen pages from Roth's original twenty.

Chapter VII is concerned with "almost all" type results. The author restricts himself to the question of the solvability of the inequalities $\|q\theta_j-\alpha_j\|<\psi(q)$, $1\leqq j\leqq n$. The easier result is that if $0\leqq\psi(q)\leqq 1/2$ for all q, then these inequalities have infinitely many solutions for almost no or almost all $2n$-dimensional sets $(\theta_1, \cdots, \theta_n, \alpha_1, \cdots, \alpha_n)$, according as the series $\sum(\psi(q))^n$ converges or diverges. It is also

shown, with considerably more difficulty, that in the case $\alpha_1 = \cdots = \alpha_n = 0$, the additional hypothesis that ψ is monotone decreasing guarantees that the inequalities in question have infinitely many solutions for almost no or almost all sets $(\theta_1, \cdots, \theta_n)$, according as the above series converges or diverges. This hypothesis is weaker than that in the similar theorem of Khintchine.

In the final chapter the Pisot-Vijayaraghavan (PV) numbers are studied. These are the algebraic integers $\alpha > 1$ all of whose conjugates except α itself lie in the open disk $|z| < 1$. It is easy to see, by considering the trace of α^n, that $\|\alpha^n\|$ approaches zero as $n \to \infty$ if α is a PV number. Pisot showed, conversely, that if $\alpha > 1$ is algebraic and $\lambda \neq 0$ is real, and if $\|\lambda\alpha^n\| \to 0$, then α is a PV number. Moreover, he showed that if $\alpha > 1$ is real, and if $\sum \|\lambda\alpha^n\|^2 < \infty$, then α is algebraic and therefore a PV number. (It is an open question whether $\|\lambda\alpha^n\| \to 0$ implies that α is algebraic.) Salem obtained the unexpected result that the set of PV numbers is closed. Proofs are given here of these three theorems; they are much simpler than the original proofs, although similar in conception.

The book closes with three appendices giving necessary tools from linear algebra and geometry of numbers, and a bibliography of papers mentioned.

W. J. LeVeque

Irrational numbers. By Ivan Niven. Carus Monograph no. 11: New York, Wiley, 1956. xii+164. $3.00.

This most recent in a series of distinguished monographs is outstanding in organization, in clarity, and in choice of material. The book, which begins with "the preponderance of irrationals," and which closes, in Chapter X, with a proof of the Gelfond-Schneider theorem, is an admirable fulfillment of the author's purpose: "an exposition of some central results on irrational numbers . . . the main emphasis [being] on those aspects . . . commonly associated with number theory and Diophantine approximations."

The topics are arranged, in general, in order of difficulty, with the result that some of the theorems in the early part of the book are subsumed under stronger theorems later. This organization seems to have real pedagogical value. The same sort of organization is followed, to some extent, within each chapter; for example, a theorem on the uniform distribution of a sequence of irrationals is first proved by use of results on continued fractions (one of the few cases where appeal is made to the material of an earlier chapter), and then ob-

tained again as a corollary of Weyl's method of trigonometric sums (the needed theorem about a Fourier series being stated and reference given for its proof).

In addition to the topics already mentioned, and to the basic subjects which would be expected (algebraic and transcendental numbers, approximation of irrationals by rationals), there are two lengthy chapters, one on Normal Numbers and another on The Generalized Lindemann Theorem. At the end of each chapter there is a set of Notes elaborating the material. A List of Notation and a Glossary supplement the Index.

Perhaps 164 pages is a natural limit for a monograph of this type, but it seems a shame that the book isn't 20 pages longer. The addition would permit the amplification of the all-too-brief Notes (as it is, Roth's result on the approximation of algebraic numbers is passed off in two sentences) and the incorporation of exercises. True, the interested reader is given ample references for further study, but the book would be much more nearly self-contained if it were slightly longer.

It may be appropriate in this review to remark that Niven smooths the way for his readers by avoiding two difficulties:

(1) *The Roadblock.* It is not uncommon to come upon a statement in print and to ask oneself, "How does he know *that* ?" The obedient and obstinate reader stops short and does his best to justify the puzzling statement, convinced that to do so should be simple—for hasn't the author thought it so obvious that he hasn't considered an explanation necessary? Reading on, one discovers that the author devotes the next page and a half to the justification. If every author would develop a style which makes a clear distinction between statements which are "obvious" and those which he proposes to prove in the sequel, readers would have an easier time.

(2) *The Hill-and-Valley.* A reader sometimes finds that he needs help in filling gaps in an exposition. Then he may ask, "Why did the author leave this to me, while, three pages back, he gave all the details of an argument that I found trivially simple without his help?" The problem of uniformity or evenness of exposition is essentially unsolvable, because of the well-known fact that one man's Mede is another man's Persian; but the conscientious expositor can at least try out his presentation on several victims before publishing it.

It is a pleasure to report that Niven has done a masterful job of exposition.

R. A. Rosenbaum

Statistical analysis of stationary time series. By Ulf Grenander and Murray Rosenblatt. John Wiley and Sons, New York, 1957. 300 pp. $11.00.

This book treats the following problem: A process $y_t = x_t + m_t$, $t = 0, \pm 1, \pm 2, \cdots$, where x_t is a stationary (or sometimes a wide-sense stationary) process and m_t a non-random sequence, is observed over a finite set $t = 0, 1, \cdots, n$ of time points. It is desired to establish certain estimates, tests, confidence regions, etc. on various structural properties of y_t, under various degrees of ignorance concerning them; e.g., to estimate the structure of m_t when it is specified except for the value of several parameters, to estimate and find confidence intervals for the spectrum of x_t when m_t is known, etc. Also, the problems of predicting, interpolating and filtering of y_t are considered.

The avowed purpose of the book is to direct the theoretical statistician's attention "to an approach to time series analysis that is essentially different from most techniques used by time series analysts in the past" and to "present a unified treatment of methods that are being used increasingly in the physical sciences and technology." The authors have strived, with good success, to make the presentation rigorous and mathematically appealing. However, its main attraction probably will be to users who are concerned with applications, as the authors have not hestitated to sacrifice completeness and generality for that which could be explicitly calculated.

The subject is still in its formative stages and, naturally, it would be unrealistic to hope for a theory as complete as that which obtains in the standard theory when the x_t are independent and identically distributed. Indeed the authors restrict the program considerably in considering only quadratic loss functions (so that much of the material is, following the geometrical treatment of Kolmogoroff, a direct consequence of the theory of unitary operators in Hilbert space), in using large sample theory ($n \to \infty$) exclusively, and in prescribing that various estimators, testing functions, and the like be linear or quadratic, etc. It would seem, thus, that statistically the art of time series is about at the stage where classical statistics was, say roughly, 30 years ago.

In the opinion of the reviewer, it would have been instructive to have formulated, in all its complexity, the problem in the general decision theory format, and to have made some attempt to assess the consequences of the simplifications dictated by analytical expediency. Indeed, a few cases have been worked out—e.g. Wald's

sequential minimax theory for the drift in an additive process—which might have been mentioned.

The material has been carefully planned and presented, and the proofs are neat and compact. Generous use has been made of outside references to most of the more delicate points, and for many of the applications. There is a set of problems at the end of the book, of a wide range of difficulty. There are a number of more or less easily rectifiable misprints.

The following is a reproduction of the table of contents: Chapter 1, Stationary Stochastic Processes and their Representation; Chapter 2, Statistical Questions when the Spectrum is Known; Chapter 3, Statistical Analysis of Parametric Models; Chapter 5, Applications; Chapter 6, Distribution of Spectral Estimates; Chapter 7, Problems in Linear Estimation; Chapter 8, Assorted Problems.

DONALD A. DARLING

Einführung in die Theorie der Differentialgleichungen im reellen Gebiet, by Ludwig Bieberbach, Berlin, Göttingen, Heidelberg, Springer-Verlag, 1956. 8+281 pp. DM 29.80. Bound DM 32.80.

This book, volume 83 in the Grundlehren series, has its genesis in the author's *Theorie der Differentialgleichungen*, which appeared as volume 6 in the same series in 1923. This earlier work had a third edition (1930), and was reprinted by Dover in 1944. Those chapters having to do with differential equations in the complex plane (which included material on analytic equations, regular and irregular singular points) have been expanded into a separate volume, *Theorie der gewöhnlichen Differentialgleichungen*, which appeared as volume 66 of the Grundlehren series. The present volume is an amplification and updating of the remaining chapters of the 1930 work. It is intended as an introduction to the subject of differential equations.

The book is divided into six sections 0–5. The introductory section 0 considers the single equation $dy/dx=f(x, y)$, and by various examples the questions of existence, uniqueness, and the behavior of solutions are posed. The section ends with a proof of the existence and uniqueness theorems assuming f satisfies a Lipschitz condition. Section 1 is an extensive treatment of existence and uniqueness results. It is much more detailed than the corresponding material in the 1930 book (56 pages to 27 pages), and for systems the author introduces vector and matrix notation. The equation $dy/dx=f(x, y)$, where f is continuous, and $|f(x, y)| \leqq M|y|+N$ on $a \leqq x \leqq b$, $|y|<\infty$, is considered. The existence theorem, using the polygonal approximations and the Ascoli lemma, is proved. Uniqueness results of the Osgood

and Nagumo variety are given, and a discussion of the maximum and minimum solutions (in cases of non-uniqueness) is given. Other topics treated in this section are continuity in f, initial conditions, and parameters, and linear systems and nth order equations.

The next section 2 on elementary integration methods, and the Runge-Kutta numerical method, is essentially the material in Chapter I, and §7, Chapter II, of the author's 1930 book. The case of a first order system of linear equations with constant coefficients is worked out in detail only for the case of a system of two equations. With very little extra labor the important general case could have been considered.

Section 3 is devoted to a study of autonomous systems close to linear ones. Critical points and the Poincaré-Bendixson theorem are first treated. The behavior of the solutions of the system $dx/dt = ax+by$, $dy/dt=cx+dy$, $ab-bc\neq 0$, is obtained by looking at the six canonical forms. Then the perturbed system $dx/dt=ax+by+p(x, y)$, $dy/dt=cx+dy+q(x, y)$, with $p, q\in C$, $p, q=o(r)$ $(r^2=x^2+y^2)$ near $(0, 0)$ is treated in great detail. It is shown in each of the six cases just what is required in order to guarantee that the behavior near $(0, 0)$ is dominated by the linear terms. Many examples are given. In the 1930 book it was assumed that p, q were analytic beginning with second degree terms. The Bendixson result on the system $dx/dt = P(x, y)+p(x, y)$, $dy/dt=Q(x, y)+q(x, y)$, with P, Q homogeneous polynomials of degree $m\geqq 1$, $yP-xQ\not\equiv 0$, $p, q=o(r^m)$, is given. New material includes detailed results on the van der Pol, Liénard equatons, and the generalization due to Levinson and Smith, the damped pendulum, as well as some results, for systems of more than two equations, on boundedness and asymptotic stability.

Boundary-value problems for second order equations receive attention in section 4. Much of this appeared in the 1930 book. In particular, the existence of eigenvalues for separated boundary conditions is shown using the Prüfer method, which depends on the oscillation, separation, and comparison theorems. The asymptotic form of the eigenfunctions and eigenvalues is given. In a book of this size it is too bad that a general treatment of regular eigenvalue problems for arbitrary nth order linear equations with self-adjoint boundary conditions could not have been included. An analysis of the periodic solutions of $\ddot{x}+a\dot{x}+bx=p(t)$, a, b constant, p periodic, is given in detail. The geometry of the solutions of the Duffing and Ricatti equations is treated.

Section 5 is a short treatment of partial differential equations of the first order, and this is essentially the same as in the 1930 work. Omit-

ted is a discussion of systems of n equations, but the case $n=2$ is considered. Another omission is an introduction to the study of the Laplace, wave, and heat equations.

With the exceptions noted above, this book should serve as an excellent introduction to the study of differential equations on the line. The total amount of material covered is just about right for a course of one semester at the senior undergradute or first year graduate level.

Two small slips were noted. On page 64 the author states that $\exp((A+B)x)=(\exp Ax)(\exp Bx)$, where A, B are matrices. This is not in general true, but it is valid in the case $B=-A$ to which it is applied. On page 169, $b(r, t)$ in formula (3.5.2) should be a *vector* of m columns and one row, instead of an m by m matrix as stated.

EARL A. CODDINGTON

Trigonometric series. A survey by R. L. Jeffery. Canadian Mathematical Congress Lecture Series, no. 2, Section III, 1953. University of Toronto Press, Toronto, 1956. 39 pages. $2.50.

Part I is a brief sketch of the high points in the historical development of Fourier series. It is fairly standard except for a discussion of the following problem: Given a function, not necessarily Lebesgue integrable, which is representable as the sum of a trigonometric series, to determine the coefficients. The author indicates the relationship between this and the problem of reconciling the Newton and Leibnitz integrals, in the solutions given by Denjoy, Marcinkiewicz and Zygmund, Burkill, and James. Part II contains detailed proofs of the theorems on Fourier series stated in Part I, as well as a sketch of the methods of James, using his P^2-integral based on the ideas of Perron.

N. J. FINE

Vector spaces and matrices. By Robert M. Thrall and Leonard Tornheim. Wiley. 1957, 12+318 pp. $6.75.

In the preface the authors announce that "In the present textbook we have chosen to proceed simultaneously at two levels, one concrete and one axiomatic. . . . Each new property of a vector space is first discussed at one level and then at the other. . . . We feel that this dual approach has many advantages. It introduces the student to the elegance and power of mathematical reasoning based on a set of axioms and helps to bridge the gap that lies between the pre-eminence of problem solving found in most elementary undergraduate courses and the axiomatic approach that characterizes much modern research in mathematics."

It has long been understood that most courses in matrices have treated the subject in a way that has little relation to the way most working mathematicians think about the topic. More emphasis should be placed on abstract vector spaces and linear transformations and less on the formalisms of the rectangular arrays. It is these ideas rather than the arrays that the student will find so useful in the study of modern mathematics. There exist very good books which provide this emphasis. But all too often students do not take easily to these abstractions and the program outlined by the authors is intended to ease their introduction to such concepts.

While the authors' dual approach is novel in a textbook it is not novel to teaching. Most instructors complement whatever text they are using with their own lectures. Thus it would seem that the authors' approach is sound and that the most serious question concerns the execution of their program. In this they succeed remarkably well, for the book retains a unity and drive toward clearly stated goals despite the seeming burden of parallel discussions.

Actually, the dual program described in the preface is carried out in full detail only in the early chapters. In the later chapters concrete and axiomatic discussions do appear, but there is not the almost self-conscious use of both at each point that characterizes the early chapters. This is really necessary, for otherwise the book would become absolutely unwieldy in size. Furthermore, it has the advantage that the student is led to do more and more thinking for himself as the work progresses.

The first seven chapters cover the material conventionally covered in a semester course in matrices. There is enough material in them that undoubtedly some omissions will be necessary in a one semester course. These seven chapters include: 1. *Vector spaces*, 2. *Linear transformations and matrices*, 3. *Systems of linear equations*, 4. *Determinants*, 5. *Equivalence relations and canonical forms*, 6. *Functions of vectors (bilinear and quadratic forms)*, 7. *Orthogonal and unitary equivalence*. The pace of the development is accelerated through this part of the book. In the early chapters almost no detail is omitted while in the later chapters many proofs are left to the student. This is very good practice for the student, but it also has the effect of de-emphasizing some important topics, particularly the diagonalization of normal matrices.

The last four chapters are presented on a much more sophisticated level and contain some of the most unusual features of the book. Chapter 8, *Structure of polynomial rings*, contains the decomposition theory for algebras for the case of a polynomial ring over a field

modulo an arbitrary fixed polynomial. The beginning student will probably find this chapter tough going, but it is well worth the effort. Chapter 9, *Equivalence of matrices over a ring*, contains the theory of invariant factors. Chapter 10, *Similarity of matrices*, contains the theory of normal forms under similarity transformations in the general case. It is always troublesome to treat this case fully. Here the real effort is invested in Chapter 8 in developing the decomposition theory. This does not make the overall development any easier, if that were the only goal. But it does have the advantage of introducing the student to some of the most important ideas of modern algebra and gives a concrete application of these ideas. Students in applied areas who take courses in matrices are usually not interested in the contents of these chapters in any form. Thus it does seem to be a good idea to cast this material in a form that will have special value for a student whose principal interest is mathematics. Chapter 11, *Linear inequalities*, contains some applications of independent interest, the minimax theorem of game theory and the equivalence of the linear programming problem with the dual linear programming problem. The proofs of these theorems depend less on the previous ten chapters than they do on some elementary topological notions.

The authors place themselves in the camp of those who prefer to write the functional notation with the symbol for the function on the right, i.e., $(x)T$ instead of $T(x)$. Since the two notations are anti-isomorphic, it is purely a matter of preference. But the authors write linear equations in the form $\sum_{\nu=1}^{n} a_{i\nu}x_\nu = b_i$ so that in matrix notation a system of equations takes the form $AX = B$. Thus the linear transformation taking X into AX is represented not by A but by its transpose. This awkwardness is avoided by speaking of transformations on spaces of row vectors and column vectors, but the connection between the concrete and the abstract is not as clear here as it should be.

The book is written with great care for accuracy and generality. But in some places the effort to obtain generality with precision results in discussions and notations which are unnecessarily cumbersome. Chapter 3 is that part of the book which suffers most from cumbersome generality. Here discussion of the row-echelon form under proper elementary row operations is alternated with the discussion of the corresponding problem under proper and improper row operations. The two forms differ by some rows of zeros and the confusion of two parallel discussions is not worth the results obtained.

Generally, the exposition is excellent. Objectives are always clearly stated and there is never doubt as to the purpose of any line of de-

velopment. To assist the beginning student many arguments, particularly mathematical induction arguments, are given in expansive detail. But the authors do this purely for the purpose of instruction and avoid adopting detailed argument as a stylistic burden. Despite taking on this necessary teaching burden the broad outline of the material remains clear. In addition the authors include much discussion of an intuitive nature. There is much to help the student acquire a "feel" for the subject.

In the overall view this book is an exceptionally good one. It is most suitable as a textbook for a better-than-average course that concerns itself with concepts at least as much as with manipulation. The better students will find much in it to stimulate their interest. The book will not appeal to the student impatient with anything that requires understanding as distinct from mere facility. But even the student most narrowly interested only in applications should find the material in the first seven chapters accessible and the viewpoint presented of definite value. The emphasis is on facility through understanding rather than facility through drill. Because the coverage of the material is quite complete with ample explanations, illustrations, and exercises, it is also a good book for the individual student learning on his own. In fact, the book is versatile and suitable for use in a wide array of circumstances and for a wide variety of purposes. It should become a standard in the field.

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